NEW WEIGHTED MULTILINEAR OPERATORS AND COMMUTATORS OF HARDY-CESÀRO TYPE∗

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Abstract This paper deals with a general class of weighted multilinear Hardy-Cesàro operators that acts on the product of Lebesgue spaces and central Morrey spaces. Their sharp bounds are also obtained. In addition, we obtain sufficient and necessary conditions on weight functions so that the commutators of these weighted multilinear Hardy-Cesàro operators (with symbols in central BMO spaces) are bounded on the product of central Morrey spaces. These results extends known results on multilinear Hardy operators.

Key words Hardy-Cesàro operators; Hardy’s inequality; multilinear Hardy operator; weighted Hardy-Littlewood averages

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1 Introduction

The Hardy integral inequality and its variants play an important role in various branches of analysis such as approximation theory, differential equations, theory of function spaces etc (see [4, 10, 18, 20, 29, 30] and references therein). The classical Hardy operator, its variants and extensions were appeared in various papers (refer to [4, 5, 13–15, 20, 21, 29–31] for surveys and historical details about these different aspects of the subject). On the other hand, the study of multilinear operators is not motivated by a mere quest to generalize the theory of linear operators but rather by their natural appearance in analysis. Coifman and Meyer in their pioneer work in the 1970s were one of the first to adopt a multilinear point of view in their study of certain singular integral operators, such as the Calderón commutators, paraproducts, and pseudodifferential operators.

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Let $\psi : [0,1] \to [0,\infty)$ be a measurable function. The weighted Hardy operator $U_\psi$ is defined on all complex-valued measurable functions $f$ on $\mathbb{R}^d$ as

$$U_\psi f(x) = \int_0^1 f(tx) \psi(t) dt.$$ (1.1)

When $\psi = 1$, this operator is reduced to the usual Hardy operator $S$ defined by $Sf(x) = \frac{1}{x} \int_0^x f(t) dt$. Results on the boundedness of $U_\psi$ on $L^p(\mathbb{R}^d)$ were first proved by Carton-Lebrun and Fosset [5]. Under certain conditions on $\psi$, the authors [5] found that $U_\psi$ is bounded from $\text{BMO}(\mathbb{R}^d)$ into itself. Furthermore, $U_\psi$ commutes with the Hilbert transform in the case $n = 1$ and with certain Calderón-Zygmund singular integral operators (and thus with the Riesz transform) in the case $n \geq 2$. A deeper extension of the results obtained in [5] was due to Jie Xiao [31].

**Theorem 1.1** (see [31]) Let $1 < p < \infty$ and $\psi : [0,1] \to [0,\infty)$ be a measurable function. Then, $U_\psi$ is bounded on $L^p(\mathbb{R}^d)$ if and only if

$$\int_0^1 t^{-n/p} \psi(t) dt < \infty.$$ (1.2)

Furthermore,

$$\|U_\psi\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \int_0^1 t^{-n/p} \psi(t) dt < \infty.$$ (1.3)

Theorem 1.1 implies immediately the following celebrated integral inequality, due to Hardy [20]

$$\|Sf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \|f\|_{L^p(\mathbb{R})}.$$ (1.4)

For further applications of Theorem 1.1, for examples the sharp bounds of classical Riemann-Liouville integral operator, see [13, 14]. Recently, Chuong and Hung [7] introduced a more general form of $U_\psi$ operator as follows.

$$U_{\psi,s} f(x) = \int_0^1 f(s(t)x) \psi(t) dt.$$ (1.5)

Here $\psi : [0,1] \to [0,\infty)$, $s : [0,1] \to \mathbb{R}$ are measurable functions and $f$ is a measurable complex valued function on $\mathbb{R}^d$. The authors in [7] obtained sharp bounds of $U_{\psi,s}$ on weighted Lebesgue and BMO spaces, where weights are of homogeneous type. They also obtained a characterization on weight functions so that the commutator of $U_{\psi,s}$, with symbols in BMO, is bounded on Lebesgue spaces. Moreover, the authors in [7] proved the boundedness on Lebesgue spaces of the following operator

$$\mathcal{H}_{\psi,s} f(x) = \int_0^\infty f(s(t)x) \psi(t) dt.$$ (1.6)

When $d = 1$ and $s(t) = \frac{1}{t}$, then $\mathcal{H}_{\psi,s}$ reduces to the classical weighted Hausdorff operator (see [25, 26] and references therein).

Very recently, the multilinear version of $U_\psi$ operators has been introduced by Fu, Gong, Lu and Yuan [14]. They defined the weighted multilinear Hardy operator as

$$U^m_\psi (f_1, \cdots, f_m) = \int_{0<t_1,\cdots,t_m<1} \left( \prod_{i=1}^m f_i (t_i x) \right) \psi(t_1, \cdots, t_m) dt_1 \cdots dt_m.$$ (1.6)
As showed in [14], when $d = 1$ and
\[ \psi(t_1, \cdots, t_m) = \frac{1}{\Gamma(\alpha) [(1 - t_1, \cdots, 1 - t_m)]^{m-\alpha}}, \]
where $\alpha \in (0; m)$, then
\[ U_{\psi}^m(f_1, \cdots, f_m)(x) = |x|^\alpha f_{\alpha}^m(f_1, \cdots, f_m)(x). \]
The operator $I_{\alpha}^m$ turns out to be the one-sided analogous to the one-dimensional multilinear Riesz operator $J_{\alpha}^m$ studied by Kenig and Stein [23], where
\[ J_{\alpha}^m(f_1, \cdots, f_m)(x) = \int_{t_1, \cdots, t_m \in \mathbb{R}} \frac{\prod_{k=1}^{m} f_k(t_k)}{|(x - t_1, \cdots, x - t_m)|^{m-\alpha}} dt_1 \cdots dt_m. \]
In [14, 17], the authors obtain the sharp bounds of $U_{\psi}^m$ on the product of Lebesgue spaces, Herz spaces, Morrey-Herz spaces and central Morrey spaces. They also proved sufficient and necessary conditions of the weight functions so that the commutators of $U_{\psi}^m$, with symbols in central BMO spaces, are bounded on the product of central Morrey spaces.

Motivated from [7, 13, 14, 17, 21], this paper aims to study the boundedness of a more general multilinear operator of Hardy type as follows.

**Definition 1.2** Let $m, n \in \mathbb{N}$, $\psi : [0, 1]^n \to [0, \infty)$, $s_1, \cdots, s_m : [0, 1]^n \to \mathbb{R}$. Given $f_1, \cdots, f_m : \mathbb{R}^d \to \mathbb{C}$ be measurable functions, we define the weighted multilinear Hardy-Cesàro operator $U_{\psi, s}^{m, n}$ by
\[ U_{\psi, s}^{m, n}(f_1, \cdots, f_m)(x) := \int_{[0, 1]^n} \left( \prod_{k=1}^{m} f_k(s_k(t)x) \right) \psi(t) dt, \quad (1.7) \]
where $s = (s_1, \cdots, s_m)$.

A multilinear version of $\mathcal{H}_{\psi, s}$ can be defined as
\[ \mathcal{H}_{\psi, s}^{m, n}(f_1, \cdots, f_m)(x) := \int_{\mathbb{R}_+^n} \left( \prod_{k=1}^{m} f_k(s_k(t)x) \right) \psi(t) dt, \quad (1.8) \]
where $\mathbb{R}_+$ is the set of all positive real numbers.

It is obviously that, when $n = m$ and $s_k(t) = t_k$, $U_{\psi, s}^{m, n}$ is reduced to $U_{\psi}^m$ as defined in (1.5). The main aim of the paper is to establish the sharp bounds of $U_{\psi, s}^{m, n}$ and $\mathcal{H}_{\psi, s}^{m, n}$ on the product of weighted Lebesgue spaces and weighted central Morrey spaces, with weights of homogeneous types. In addition, we prove sufficient and and necessary conditions of the weight functions so that commutators of such weighted multilinear Hardy-Cesàro operators (with symbols in $\lambda$-central BMO space) are bounded on the product of central Morrey spaces. Since $U_{\psi, s}^{m, n}$ is trivially more general than known operators $U_{\psi}, U_{\psi}^m$, our results can be used to recover main results of [7, 13, 14, 31].

Throughout the whole paper, the letter $C$ will indicate an absolute constant, probably different at different occurrences. With $\lambda \in \mathcal{X}$ we will denote the characteristic function of a set $E$ and $B(x, r)$ will be a ball centered at $x$ with radius $r$. With $|A|$ we will denote the Lebesgue measure of a measurable set $E$, and $E^c$ will be the set $\mathbb{R}^d \setminus E$. With $\omega(E)$ we will denote by $\int_E \omega(x) dx$ and $S_d$ will be the unit ball $\{ x \in \mathbb{R}^d : |x| = 1 \}$. 
2 Notations and Definitions

Throughout this paper, $\omega(x)$ will be denote a nonnegative measurable function on $\mathbb{R}^d$. Let us recall that a measurable function $f$ belongs to $L^p_{\omega}(\mathbb{R}^d)$ if

$$\|f\|_{L^p_{\omega}} = \left( \int_{\mathbb{R}^d} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty. \quad (2.1)$$

The weighted BMO space $\text{BMO}(\omega)$ is defined as the set of all functions $f$, which are of bounded mean oscillation with weight $\omega$, that is,

$$\|f\|_{\text{BMO}(\omega)} = \sup_B \frac{1}{\omega(B)} \int_B |f(x) - f_{B,\omega}| \omega(x) \, dx < \infty, \quad (2.2)$$

where supremum is taken over all the balls $B \subset \mathbb{R}^d$. Here we use the standard notation $\omega(B) = \int_B \omega(x) \, dx$ and $f_{B,\omega}$ is the mean value of $f$ on $B$ with weight $\omega$:

$$f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x) \omega(x) \, dx.$$ 

The case $\omega \equiv 1$ of (2.2) corresponds to the class of functions of bounded mean oscillation of John and Nirenberg [22]. We observe that $L^{\infty}(\mathbb{R}^d) \subset \text{BMO}(\omega)$.

Next we recall the definition of Morrey spaces. It is well-known that Morrey spaces are useful to study the local behavior of solutions to second-order elliptic partial differential equations and the boundedness of Hardy-Littlewood maximal operator, the fractional integral operators, singular integral operators (see [1, 6, 24]). We notice that the weighted Morrey spaces were first introduced by Komori and Shirai [24], where they used them to study the boundedness of some important classical operators in harmonic analysis like as Hardy-Littlewood maximal operator, Calderón-Zygmund operators.

**Definition 2.1** Let $\lambda \in \mathbb{R}$, $1 \leq p < \infty$ and $\omega$ be a weight function. The weighted Morrey space $L^{p,\lambda}_{\omega}(\mathbb{R}^d)$ is defined by the set of all locally $p$-integrable functions $f$ satisfying

$$\|f\|_{L^{p,\lambda}_{\omega}(\mathbb{R}^d)} = \sup_{a \in \mathbb{R}^d, R > 0} \left( \frac{1}{\omega(B(a, R))^{1+\lambda/p}} \int_{B(0,R)} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty. \quad (2.3)$$

The spaces of bounded central mean oscillation $\text{CMO}^q(\mathbb{R}^d)$ appear naturally when considering the dual spaces of the homogeneous Herz type Hardy spaces and were introduced by Lu and Yang (see [27]). The relationships between central BMO spaces and Morrey spaces were studied by Alvarez, Guzmán-Partida and Lakey [2]. Furthermore, they introduced $\lambda$-central BMO spaces and central Morrey spaces as follows.

**Definition 2.2** Let $\lambda \in \mathbb{R}$ and $1 < p < \infty$. The weighted central Morrey space $\dot{B}^{p,\lambda}_{\omega}(\mathbb{R}^d)$ is defined by the set of all locally $p$-integrable functions $f$ satisfying

$$\|f\|_{\dot{B}^{p,\lambda}_{\omega}(\mathbb{R}^d)} = \sup_{R > 0} \left( \frac{1}{\omega(B(0,R))^{1+\lambda/p}} \int_{B(0,R)} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty. \quad (2.4)$$

Obviously, $\dot{B}^{p,\lambda}_{\omega}(\mathbb{R}^d)$ is a Banach space and one can easily check that $\dot{B}^{p,\lambda}_{\omega}(\mathbb{R}^d) = \{0\}$ if $\lambda < -\frac{1}{q}$. Similar to the classical Morrey space, we only consider the case $-1/p \leq \lambda < 0$. 

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Theorem 3.1 Let \( s_1, \ldots, s_m : [0, 1]^n \to \mathbb{R} \) be measurable functions such that for every \( k = 1, \ldots, m \) then \( |s_k(t_1, \ldots, t_n)| \geq \min\{\tau_1^\beta, \ldots, \tau_d^\beta\} \) almost everywhere \((t_1, \ldots, t_n) \in [0, 1]^n\), \( k = 1, \ldots, m \), for some \( \beta > 0 \). Let \( 1 \leq p_k < \infty \), \( k = 1, \ldots, m \) and \( \omega = (s_1, \ldots, s_m) \). Assume that \((\omega_1, \ldots, \omega_m)\) satisfies \( W_{1\alpha} \) condition. Then \( U_{\omega, \pi}^{m, n} \) is bounded from \( L^p_{\omega_1} (\mathbb{R}^d) \times \cdots \times L^p_{\omega_m} (\mathbb{R}^d) \) to \( L^p_{\omega} (\mathbb{R}^d) \) if and only if

\[
\mathcal{A} = \int_{[0, 1]^n} \left( \prod_{k=1}^m |s_k(t)|^{\frac{d-p_k}{p_k}} \right) \psi(t) dt < \infty. \tag{3.4}
\]

Furthermore,

\[
\left\| U_{\omega, \pi}^{m, n} (f_1, \ldots, f_m) \right\|_{L^p_{\omega} (\mathbb{R}^d)} \leq \left( \int_{\mathbb{R}^d} \left( \int_{[0, 1]^n} \prod_{k=1}^m |f_k(s_k(t)x)| \psi(t) dt \right)^p \omega(x) dx \right)^{1/p} \leq \int_{[0, 1]^n} \left( \int_{\mathbb{R}^d} \prod_{k=1}^m |f_k(s_k(t)x)|^p \omega(x) dx \right)^{1/p} \psi(t) dt \leq \int_{[0, 1]^n} \prod_{k=1}^m \left( \int_{\mathbb{R}^d} \prod_{k=1}^m |f_k(s_k(t)x)|^{p_k} \omega_k(x) dx \right)^{1/p_k} \psi(t) dt.
\]

The last inequality implies that \( U_{\omega, \pi}^{m, n} \) is bounded from \( L^p_{\omega_1} (\mathbb{R}^d) \times \cdots \times L^p_{\omega_m} (\mathbb{R}^d) \) to \( L^p_{\omega} (\mathbb{R}^d) \) and

\[
\left\| U_{\omega, \pi}^{m, n} \right\|_{L^p_{\omega_1} (\mathbb{R}^d) \times \cdots \times L^p_{\omega_m} (\mathbb{R}^d) \to L^p_{\omega} (\mathbb{R}^d)} \leq \mathcal{A}. \tag{3.6}
\]

In order to prove the converse of the theorem, we first need the following lemma.

Lemma 3.2 Let \( \omega \in W_{1\alpha}, \alpha > -d \) and \( \varepsilon > 0 \). Then the function

\[
f_{p, \varepsilon}(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ |x|^{-\frac{d+\alpha}{p}} & \text{if } |x| > 1, \end{cases}
\]

belongs to \( L^p_{\omega} (\mathbb{R}^d) \) and \( \|f_{p, \varepsilon}\|_{p, \omega} = (\frac{\omega(\mathbb{S}_d)}{p^\varepsilon})^{1/p} \).

Since the proof of the lemma is straightforward, we omit it. Now, let \( \varepsilon \) be an arbitrary positive number and for each \( k = 1, \ldots, m \) we set \( \varepsilon_k = \frac{\omega_k(\mathbb{S}_d)}{p_k} \) and

\[
f_{p_k, \varepsilon_k}(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ |x|^{-\frac{d+\alpha_k}{p_k} - \varepsilon_k} & \text{if } |x| > 1. \end{cases}
\]

Lemma 3.2 implies that \( f_{p_k, \varepsilon_k} \in L^p_{\omega_k} (\mathbb{R}^d) \) and

\[
\|f_{p_k, \varepsilon_k}\|_{p_k, \omega_k} = \left( \frac{\omega_k(\mathbb{S}_d)}{p_k \varepsilon_k} \right)^{1/p_k} \left( \frac{\omega_k(\mathbb{S}_d)}{p \varepsilon} \right)^{1/p_k} > 0,
\]
for each $k = 1, \ldots, m$. For each $x \in \mathbb{R}^d$ which $|x| \geq 1$, let

$$S_x = \bigcap_{k=1}^{m} \{ t \in [0,1]^n : |s_k(t)x| > 1 \}.$$  

From the assumption $|s_k(t_1, \ldots, t_n)| \geq \min \{ t_1^\beta, \ldots, t_n^\beta \}$ a.e. $t = (t_1, \ldots, t_n) \in [0,1]^n$, there exists a null subset $E$ of $[0,1]^n$ so that $S_x$ contains $\{1/|x|^{1/\beta}, 1\} \setminus E$. From (2.1), we have

$$\|U_{\psi, \varphi}^{m,n}(f_{p_1, \varepsilon_1}, \ldots, f_{p_m, \varepsilon_m})\|^p_{L^p_\psi(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| \int_{[0,1]^n} \left( \prod_{k=1}^{m} f_{k}(s_k(t)x) \right) \psi(t) dt \right|^p \omega(x) dx$$

$$= \int_{\mathbb{R}^d} \left| \int_{[0,1]^n} \left( \prod_{k=1}^{m} f_{k}(s_k(t)x) \right) \psi(t) dt \right|^p \omega(x) dx$$

$$\geq \int_{|x| \geq \varepsilon^{-\beta}} |x|^{-\beta} \omega(x) \left| \int_{[0,1]^n} \left( \prod_{k=1}^{m} |s_k(t)|^{-\frac{d+n_k}{p_k}} \psi(t) dt \right) \right|^p \omega(x) dx$$

$$\geq \varepsilon \beta \left( \int_{|x| \geq \varepsilon} |x|^{-\beta} \omega(x) dx \right) \left| \int_{[0,1]^n} \left( \prod_{k=1}^{m} |s_k(t)|^{-\frac{d+n_k}{p_k}} \psi(t) dt \right) \right|^p \omega(\varphi)$$

This implies that

$$\left\| U_{\psi, \varphi}^{m,n} \right\|^p_{L^p_\psi(\mathbb{R}^d) \times \ldots \times L^p_{\varphi,m}(\mathbb{R}^d) \rightarrow L^p_\varphi(\mathbb{R}^d)} \geq \varepsilon \beta \left( \int_{[0,1]^n} \left( \prod_{k=1}^{m} |s_k(t)|^{-\frac{d+n_k}{p_k}} \psi(t) dt \right) \right).$$

Notice that

$$|s_k(t)|^{-\varepsilon} \leq \min \{ t_1^\beta, \ldots, t_n^\beta \}^{-\varepsilon \beta} \leq \varepsilon^{-\varepsilon \beta} \rightarrow 1 \quad \text{when } \varepsilon \rightarrow 0^+,$$

Thus, letting $\varepsilon \rightarrow 0^+$ and by Lebesgue’s dominated convergence theorem, we obtain

$$\left\| U_{\psi, \varphi}^{m,n} \right\|^p_{L^p_\psi(\mathbb{R}^d) \times \ldots \times L^p_{\varphi,m}(\mathbb{R}^d) \rightarrow L^p_\varphi(\mathbb{R}^d)} \geq \mathcal{A}. \quad (3.7)$$

Combine (3.6) and (3.7), we obtain the result. $\square$

**Theorem 3.3** Let $s_1, \ldots, s_m : [0,1]^n \rightarrow \mathbb{R}$ be measurable functions such that for every $k = 1, \ldots, m$ we have $|s_k(t_1, \ldots, t_n)| \geq \min \{ t_1^\beta, \ldots, t_n^\beta \}$ almost everywhere $(t_1, \ldots, t_n) \in [0,1]^n$, $k = 1, \ldots, m$, for some $\beta > 0$. Let $1 \leq p_k < \infty$, $k = 1, \ldots, m$ and $\varphi = (s_1, \ldots, s_m)$. Assume that $(\omega_1, \ldots, \omega_m)$ satisfies $\mathcal{W}_{s_1}^\alpha$ condition. Then $\mathcal{H}_{\psi, \varphi}^{m,n}$ is bounded from $L^{p_1}_{\omega_1}(\mathbb{R}^d) \times \ldots \times L^{p_m}_{\omega_m}(\mathbb{R}^d)$ to $L^p_{\omega}(\mathbb{R}^d)$ if and only if

$$\mathcal{A}_* = \int_{\mathbb{R}^d^+} \left( \prod_{k=1}^{m} |s_k(t)|^{-\frac{d+n_k}{p_k}} \right) \psi(t) dt < \infty. \quad (3.8)$$

Furthermore,

$$\left\| U_{\psi, \varphi}^{m,n} \right\|^p_{L^p_\psi(\mathbb{R}^d) \times \ldots \times L^p_{\varphi,m}(\mathbb{R}^d) \rightarrow L^p_\varphi(\mathbb{R}^d)} = \mathcal{A}_*. \quad (3.9)$$
Theorem 3.4
Let $1 \leq p, p_k < \infty$, $\lambda, \alpha_k, \lambda_k$ be real numbers such that $-\frac{1}{p_k} \leq \lambda_k < 0$, $k = 1, \ldots, m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\lambda = \frac{d+\alpha_1}{d+\alpha} \lambda_1 + \cdots + \frac{d+\alpha_m}{d+\alpha} \lambda_m$. Let $\omega_k \in W_{\alpha_k}$ for $k = 1, \ldots, m$.

(i) If in addition
\[\left(\frac{\omega(S_d)}{d + \alpha}\right)^{\frac{1+\lambda}{p}} \geq \prod_{k=1}^{m} \left(\frac{\omega_k(S_d)}{d + \alpha_k}\right)^{\frac{1+\lambda_k}{p_k}}\] (3.10)
and
\[\mathcal{B} = \int_{[0,1]^n} \left( \prod_{k=1}^{m} \left| s_k(t) \right|^{-\frac{(d+\alpha_k+1)\lambda_k}{p_k}} \right) \psi(t) dt < \infty,\] (3.11)
then $U_{\psi, \frac{1}{\alpha}}^{m,n}$ is bounded from $\dot{B}_{\omega_1}^{p_1, \lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m}^{p_m, \lambda_m} (\mathbb{R}^d)$ to $\dot{B}_{\psi}^{p, \lambda} (\mathbb{R}^d)$. Furthermore, the operator norm of $U_{\psi, \frac{1}{\alpha}}^{m,n}$ is not greater than $\mathcal{B}$.

(ii) Conversely, if
\[\left(\frac{\omega(S_d)}{d + \alpha}\right)^{\lambda} (1 + \lambda p)^{1/p} \leq \prod_{k=1}^{m} \left(\frac{\omega(S_d)}{d + \alpha_k}\right)^{\lambda_k} (1 + \lambda_k p_k)^{1/p_k}\] (3.12)
and $U_{\psi, \frac{1}{\alpha}}^{m,n}$ is bounded from $\dot{B}_{\omega_1}^{p_1, \lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m}^{p_m, \lambda_m} (\mathbb{R}^d)$ to $\dot{B}_{\psi}^{p, \lambda} (\mathbb{R}^d)$, then $\mathcal{B}$ is finite. Furthermore, we have
\[\left\| U_{\psi, \frac{1}{\alpha}}^{m,n} \right\|_{\dot{B}_{\omega_1}^{p_1, \lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m}^{p_m, \lambda_m} (\mathbb{R}^d) \to \dot{B}_{\psi}^{p, \lambda} (\mathbb{R}^d)} \geq \mathcal{B}. \] (3.13)

Let $\omega_k \equiv 1$, we have that $\alpha_k = 0$ thus (3.10) becomes equality and (3.12) holds only when $\lambda_1 p_1 = \cdots = \lambda_m p_m$. Hence, we obtain Theorem 2.1 in [14].

Proof
Since $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, by Minkowski’s inequality and Hölder’s inequality, we see that, for all balls $B = B(0, R)$,
\[
\left(\frac{1}{\omega(B)}\right)^{1+\lambda p} \int_{B} \left| U_{\psi, \frac{1}{\alpha}}^{m,n} (f_1, \ldots, f_m) (x) \right|^p \omega(x) dx \right)^{1/p}
\leq \int_{[0,1]^n} \left( \prod_{k=1}^{m} \left| f_k (s_k(t)x) \right|^p \omega(x) dx \right)^{1/p} \psi(t) dt
\leq \int_{[0,1]^n} \left[ \prod_{k=1}^{m} \left| f_k (s_k(t)x) \right|^p \omega_k(x) dx \right]^{1/p_k} \psi(t) dt
\leq \int_{[0,1]^n} \left[ \prod_{k=1}^{m} \left| s_k(t) \right|^p \omega_k(y) dy \right]^{1/p_k} \psi(t) dt
\leq \prod_{k=1}^{m} \left\| f_k \right\|_{\dot{B}_{\psi}^{p_k, \lambda_k} (\mathbb{R}^d)} \cdot \mathcal{B}.
\]
This means that
\[\left\| U_{\psi, \frac{1}{\alpha}}^{m,n} \right\|_{\dot{B}_{\omega_1}^{p_1, \lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m}^{p_m, \lambda_m} (\mathbb{R}^d) \to \dot{B}_{\psi}^{p, \lambda} (\mathbb{R}^d)} \leq \mathcal{B}. \] (3.14)

Now we assume that (3.12) holds and $U_{\psi, \frac{1}{\alpha}}^{m,n}$ is bounded from $\dot{B}_{\omega_1}^{p_1, \lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m}^{p_m, \lambda_m}$ to $\dot{B}_{\psi}^{p, \lambda} (\mathbb{R}^d)$. Let $f_k(x) = |x|^{d+\alpha_k} \lambda_k$. Then an elementary computation shows that $f_k \in \dot{B}_{\omega_k}^{p_k, \lambda_k} (\mathbb{R}^d)$ and
\[
\left\| f_k \right\|_{\dot{B}_{\omega_k}^{p_k, \lambda_k} (\mathbb{R}^d)} = \left( \frac{d + \alpha_k}{\omega(S_d)} \right)^{\lambda_k} (1 + \lambda_k p_k)^{-1/p_k}.
\]
Thus, 
\[
\prod_{k=1}^{m} \| f_k \|_{\dot{B}_{p_k}^{\lambda_k}(\mathbb{R}^d)} = \prod_{k=1}^{m} \left( \frac{d + \alpha_k}{\omega(S_d)} \right)^{\lambda_k} \left( (1 + \lambda_k p_k)^{-1/p_k} \right) \leq \left( \frac{d + \alpha}{\omega(S_d)} \right)^{\lambda} (1 + \lambda p)^{-1/p}. 
\]
This leads us to 
\[
\| U_{\psi, \overline{\sigma}}^{m,n} (f_1, \ldots, f_m) \|_{\dot{B}_\infty^{\lambda} (\mathbb{R}^d)} = B \cdot \left( \frac{d + \alpha}{\omega(S_d)} \right)^{\lambda} (1 + \lambda p)^{-1/p} \geq B \cdot \prod_{k=1}^{m} \| f_k \|_{\dot{B}_{p_k}^{\lambda_k} (\mathbb{R}^d)}. 
\]
Therefore, 
\[
\| U_{\psi, \overline{\sigma}}^{m,n} \|_{\dot{B}_1^{\lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_m^{\lambda_m} (\mathbb{R}^d) \to \dot{L}_{\infty}^\alpha (\mathbb{R}^d)} \geq B. 
\]

From the proof of Theorem 3.4, we also obtain the similar result to \( \mathcal{H}_{\psi, \overline{\sigma}}^{m,n} \) operator.

**Theorem 3.5** Let \( 1 \leq p, p_k < \infty, \lambda, \alpha_k, \lambda_k \) be real numbers such that \( -\frac{1}{p_k} \leq \lambda_k < 0, \) \( k = 1, \ldots, m, \frac{1}{p} = \frac{1}{p_1} \cdots + \frac{1}{p_m}, \) and \( \lambda = \frac{d + m_1}{d + \alpha} \lambda_1 + \cdots + \frac{d + m_m}{d + \alpha} \lambda_m. \) Let \( \omega_k \in \mathcal{W}_{\alpha_k} \) for \( k = 1, \cdots, m. \)

(i) If in addition 
\[
\left( \frac{\omega(S_d)}{d + \alpha} \right)^{\frac{1}{p} + \frac{\lambda}{p_k}} \geq \prod_{k=1}^{m} \left( \frac{\omega_k(S_d)}{d + \alpha_k} \right)^{\frac{1 + \lambda p_k}{p_k}} 
\]
and
\[
\mathcal{B}_* = \int_{\mathbb{R}^d_+} \left( \prod_{k=1}^{m} | s_k(t) |^{\frac{d + \alpha_k}{p_k}} \right)^{\frac{1}{p_k}} \psi(t) dt < \infty, 
\]
then \( \mathcal{H}_{\psi, \overline{\sigma}}^{m,n} \) is bounded from \( \dot{B}_1^{\lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_m^{\lambda_m} (\mathbb{R}^d) \) to \( \dot{B}_\infty^{\lambda} (\mathbb{R}^d). \) Furthermore, the operator norm of \( \mathcal{H}_{\psi, \overline{\sigma}}^{m,n} \) is not greater than \( \mathcal{B}_*. \)

(ii) Conversely, if 
\[
\left( \frac{\omega(S_d)}{d + \alpha} \right)^{\lambda} (1 + \lambda p)^{1/p} \leq \prod_{k=1}^{m} \left( \frac{\omega(S_d)}{d + \alpha_k} \right)^{\lambda_k} (1 + \lambda_k p_k)^{1/p_k} 
\]
and \( \mathcal{H}_{\psi, \overline{\sigma}}^{m,n} \) is bounded from \( \dot{B}_1^{\lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_m^{\lambda_m} (\mathbb{R}^d) \) to \( \dot{B}_\infty^{\lambda} (\mathbb{R}^d), \) then \( \mathcal{B}_* \) is finite. Furthermore, we have 
\[
\| \mathcal{H}_{\psi, \overline{\sigma}}^{m,n} \|_{\dot{B}_1^{\lambda_1} (\mathbb{R}^d) \times \cdots \times \dot{B}_m^{\lambda_m} (\mathbb{R}^d) \to \dot{L}_{\infty}^\alpha (\mathbb{R}^d)} \geq \mathcal{B}_*. 
\]

### 4 Commutators of Weighted Multilinear Hardy-Cesàro Operator

We use some analogous tools to study a second set of problems related now to multilinear versions of the commutators of Coifman, Rochberg and Weiss [9]. The boundedness of commutators of weighted Hardy operators on the Lebesgue spaces, with symbols in BMO space, have been studied in [13] by Fu, Liu and Lu. Their main idea is to control the commutators of weighted Hardy operators by the weighted Hardy-Littlewood maximal operators. However this method is not easy to extend to deal with the multilinear case, since the maximal operators which are suitable to control commutator in this case are less known. In [14], the authors give an idea to replace Lebesgue spaces with central Morrey spaces and symbols are taken in central BMO spaces. By this method, we can avoid to use multilinear maximal operators.
Our purpose of this section is to apply that method in order to characterize weight functions such that the multilinear commutator generated by $U_{\psi,s}^{m,n}$, with symbols in central BMO spaces, are bounded from the product of central Morrey spaces into central Morrey spaces.

**Definition 4.1** Let $m, n \in \mathbb{N}$, $\psi : [0, 1]^n \to [0, \infty)$, $s_1, \ldots, s_m : [0, 1]^n \to \mathbb{R}$, $b_1, \ldots, b_m$ be locally integrable functions on $\mathbb{R}^d$ and $f_1, \ldots, f_m : \mathbb{R}^d \to \mathbb{C}$ be measurable functions. The commutator of weighted multilinear Hardy-Cesàro operator $U_{\psi,s}^{m,n}$ is defined as

$$U_{\psi,s}^{m,n} (f_1, \ldots, f_m) (x) := \int_{[0,1]^n} \left( \prod_{k=1}^m f_k (s_k(t)x) \right) \left( \prod_{k=1}^m (b_k(x) - b_k (s_k(t)x)) \right) \psi(t) dt. \quad (4.1)$$

In what follows, we set

$$C = \int_{[0,1]^n} \left( \prod_{k=1}^m |s_k(t)|^{(d+\alpha_k)\lambda_k} \right) \psi(t) dt, \quad (4.2)$$

$$D = \int_{[0,1]^n} \left( \prod_{k=1}^m |s_k(t)|^{(d+\alpha_k)\lambda_k} \right) \left( \prod_{k=1}^m \log |s_k(t)| \right) \psi(t) dt. \quad (4.3)$$

To state our result, let us introduce some notations. The symbol $f \lesssim g$ means that $f \leq Cg$. We denote $p, p_1, \ldots, p_m, q_1, \ldots, q_m, \lambda_1, \ldots, \lambda_m$ and $\alpha_1, \ldots, \alpha_m$ by real numbers such that $1 < p < p_k < \infty$, $1 < q_k < \infty$, $\frac{1}{p_k} < \lambda_k < 0$, $\alpha_1, \ldots, \alpha_m > -d$ and

$$\frac{1}{p} = \sum_{k=1}^m \frac{1}{p_k}, \quad \alpha = \sum_{k=1}^m p\alpha_k \left( \frac{1}{p_k} + \frac{1}{q_k} \right), \quad \lambda = \sum_{k=1}^m \frac{d + \alpha_k \lambda_k}{d + \alpha}.$$

We shall consider weight functions $\omega_k \in \mathcal{W}_{\alpha_k}, k = 1, \ldots, m$, such that

$$\prod_{k=1}^m \left( \frac{\omega_k(S_q)}{d + \alpha_k} \right)^{\frac{1 + \alpha_k q_k}{q_k p}} \lesssim \left( \frac{\omega(S_d)}{d + \alpha} \right)^{\frac{1 + \alpha p}{p}}, \quad (4.4)$$

where $\omega$ is determined by $\omega = \prod_{k=1}^m \omega_k^{p/p_k + p/q_k}$.

Our main result on the commutator $U_{\psi,s}^{m,n}$ is as follows.

**Theorem 4.2** (i) If both $C$ and $D$ are finite then for any $b = (b_1, \ldots, b_m) \in \text{CMO}_{\omega_1} \times \cdots \times \text{CMO}_{\omega_m}$ then $U_{\psi,s}^{m,n}$ is bounded from $\dot{B}_{\omega_1,\lambda_1}^p (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m,\lambda_m}^p (\mathbb{R}^d)$ to $\dot{B}_{\omega}^p (\mathbb{R}^d)$.

(ii) If for any $b = (b_1, \ldots, b_m) \in \text{CMO}_{\omega_1} \times \cdots \times \text{CMO}_{\omega_m}, U_{\psi,s}^{m,n}$ is bounded from $\dot{B}_{\omega_1,\lambda_1}^p (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m,\lambda_m}^p (\mathbb{R}^d)$ to $\dot{B}_{\omega}^p (\mathbb{R}^d)$, then $D$ is finite.

We note here that $D$ is finite is not enough to imply $C$ is finite (see [13, 14]). But it is easy to see that, if we assume in addition that for each $k = 1, \ldots, m$ such that $|s_k(t)| \geq c > 1$ for all $t \in [0, 1]^n$ or $|s_k(t)| \leq c < 1$ for all $t \in [0, 1]^n$, then $C$ is finite if and only if $D$ is finite. Thus, Theorem 4.2 implies immediately that

**Corollary 4.3** Suppose that for each $k = 1, \ldots, m$ then $|s_k(t)| \geq c > 1$ for all $t \in [0, 1]^n$ or $|s_k(t)| \leq c < 1$ for all $t \in [0, 1]^n$, for each $k = 1, \ldots, m$. Then, $U_{\psi,s}^{m,n}$ is bounded from $\dot{B}_{\omega_1,\lambda_1}^p (\mathbb{R}^d) \times \cdots \times \dot{B}_{\omega_m,\lambda_m}^p (\mathbb{R}^d)$ to $\dot{B}_{\omega}^p (\mathbb{R}^d)$ if and only if $C$ is finite.

Now we will give the proof of Theorem 4.2.

**Proof** First we assume that $C$ is finite. We shall give a details analysis for the case $m = 2$ and by similarity, the general case is proved in the same way. We denote $B = B(0, R)$ for short.
Let \( \overrightarrow{b} = (b_1, b_2) \in \text{CMO}_{\mathcal{B}_d}^m(\mathbb{R}^d) \times \text{CMO}_{\mathcal{B}_d}^n(\mathbb{R}^d) \). By Minkowski’s inequality, we have

\[
\left( \frac{1}{\omega(B)} \int_B \left| U_{\psi,s}^{2,n,b}(f_1, f_2)(x) \right|^p \omega(x) \, dx \right)^{1/p} 
\leq \left( \frac{1}{\omega(B)} \int_B \left( \int_{[0,1]^n} \left( \sum_{k=1}^2 |f_k(s_k(t)x)| \right) \psi(t) \, dt \right)^p \omega(x) \, dx \right)^{1/p} 
\leq \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \sum_{k=1}^2 |f_k(s_k(t)x)| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt 
=: I.
\]

For any \( x_i, y_i, z_i, t_i \in \mathbb{C} \) with \( i = 1, 2 \), we have the following elementary inequality

\[
\prod_{i=1}^2 |(x_i - y_i)| \leq \prod_{i=1}^2 |(x_i - z_i)| + \prod_{i=1}^2 |(y_i - t_i)| + \prod_{i=1}^2 |(z_i - t_i)| 
+ |(x_1 - z_1)(z_2 - t_2)| + |(x_2 - z_2)(z_1 - t_1)| 
+ |(x_1 - z_1)(y_2 - t_2)| + |(x_2 - z_2)(y_1 - t_1)| 
+ |(z_1 - t_1)(y_2 - t_2)| + |(z_2 - t_2)(y_1 - t_1)|.
\]

It is convenient to denote by \( b_i, \omega, B \) the integrals \( \int_B \frac{1}{\omega(B)} b_i(x) \omega(x) \, dx \) for \( i = 1, 2 \). Now applying the inequality with \( x_i = b_i(x), y_i = b_i(s_i(t)x), z_i = b_i(B), t_i = b_i(s_i(t)B) \) and using Minkowski’s inequality, we get that

\[
I \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]

where, if we set \( \mathcal{I}(x) = \prod_{k=1}^2 |f_k(s_k(t)x)| \), then

\[
I_1 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \mathcal{I}(x) \prod_{k=1}^2 |b_k(x) - b_k, \omega_k, B| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt,
\]

\[
I_2 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \mathcal{I}(x) \prod_{k=1}^2 |b_k(s_k(t)x) - b_k, \omega_k, s_k(t)B| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt,
\]

\[
I_3 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \mathcal{I}(x) \prod_{k=1}^2 |b_k, \omega_k, B - b_k, \omega_k, s_k(t)B| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt,
\]

\[
I_4 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \mathcal{I}(x) \prod_{i \neq j} \sum_{k=1}^2 |(b_i(x) - b_i, B)(b_j, B - b_j, \omega_j, s_j(t)B)| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt,
\]

\[
I_5 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \int_B \left( \mathcal{I}(x) \prod_{i \neq j} \sum_{k=1}^2 |(b_i(x) - b_i, B)(b_j, B - b_j, \omega_j, s_j(t)B)| \right) \omega(x) \, dx \right)^{1/p} \psi(t) \, dt.
\]
\[ I_6 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \left( f^p(x) \int_B \omega(x)dx \right)^{1/p} \psi(t)dt \right) \]

\[ \cdot \sum_{i,j=1,2} \left| (b_i(x) - b_i,B) \left( b_j(s_j(t)x) - b_j,\omega_j,\omega_j(t)B \right) \right|^p \omega(x)dx \]

\[ \psi(t)dt, \]

\[ I_6 = \int_{[0,1]^n} \left( \frac{1}{\omega(B)} \left( f^p(x) \int_B \omega(x)dx \right)^{1/p} \psi(t)dt \right) \]

\[ \cdot \sum_{i,j=1,2} \left| (b_i,B - b_i,s_i(B)) \left( b_j(s_j(t)x) - b_j,\omega_j,s_j(t)B \right) \right|^p \omega(x)dx \]

\[ \psi(t)dt. \]

Choose now \( p < s_1 < \infty \) and \( p < s_2 < \infty \) such that \( \frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{q_1} \) and \( \frac{1}{s_2} = \frac{1}{p_2} + \frac{1}{q_2} \). Notice that \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{p} \). Then by Hölder’s inequality and (4.4), we deduce that

\[ I_1 \lesssim \int_{[0,1]^n} \prod_{k=1}^2 \left( \frac{1}{\omega_k(B)^{1+\lambda_k}} \int_B |f_k(s_k(t)x)|^{p_k} \omega_k(x)dx \right)^{1/p_k} \]

\[ \times \prod_{k=1}^2 \left( \frac{1}{\omega_k(B)} \int_B |b_k(x) - b_k,\omega_k,B|^{q_k} \omega_k(x) \right)^{1/q_k} \psi(t)dt \]

\[ \lesssim \prod_{k=1}^2 \| b_k \|_{C_{M}^{\nu_k}} \prod_{k=1}^2 \| f_k \|_{L_{p_k,\mu_k}} \times \int_{[0,1]^n} \left( \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \right) \psi(t)dt. \]

Similarly to the estimate of \( I_1 \), we have that

\[ I_2 \lesssim \prod_{k=1}^2 \| b_k \|_{C_{M}^{\nu_k}} \prod_{k=1}^2 \| f_k \|_{L_{p_k,\mu_k}} \times \int_{[0,1]^n} \left( \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \right) \psi(t)dt. \]

Now we give the estimate for \( I_3 \). Applying Hölder’s inequality, we get that

\[ I_3 \lesssim \int_{[0,1]^n} \prod_{k=1}^2 \left( \frac{1}{\omega_k(s_k(t)B)^{1+\mu_k}} \int_{s_k(t)B} |f_k(y)|^{p_k} \omega_k(y)dy \right)^{1/p_k} \]

\[ \times \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \omega_k(B)^{\lambda_k} \prod_{k=1}^2 |b_k,\omega_k,B - b_k,\omega_k,s_k(t)B| \psi(t)dt \]

\[ \lesssim \prod_{k=1}^2 \| f_k \|_{L_{p_k,\mu_k}} \int_{[0,1]^n} \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \prod_{k=1}^2 |b_k,\omega_k,B - b_k,\omega_k,s_k(t)B| \psi(t)dt. \]

Notice that \( [0,1]^n \) is the union of pairwise disjoint subsets \( S_{\ell_1,\ell_2} \), where

\[ S_{\ell_1,\ell_2} = \bigcap_{i=1}^2 \{ t \in [0,1]^n : 2^{-\ell_1} - 2^{-\ell_1} < |s_i(t)| \leq 2^{-\ell_1} \}. \]

Thus we obtain that

\[ I_3 \lesssim \prod_{k=1}^2 \| f_k \|_{L_{p_k,\mu_k}} \sum_{\ell_1,\ell_2} \int_{S_{\ell_1,\ell_2}} \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \psi(t) \]

\[ \times \prod_{k=1}^2 \left( \sum_{j=0}^{\ell_k} |b_k,\omega_k,2^{-j-1}B - b_k,\omega_k,2^{-j}B| + |b_k,\omega_k,2^{-\ell_k-1}B - b_k,\omega_k,s_k(t)B| \right) dt \]

\[ \lesssim \prod_{k=1}^2 \| f_k \|_{L_{p_k,\mu_k}} \sum_{\ell_1,\ell_2} \int_{S_{\ell_1,\ell_2}} \prod_{k=1}^2 |s_k(t)|^{(d+\alpha_k)\lambda_k} \prod_{k=1}^2 (\ell_k + 2) |b_k|_{C_{M}^{\nu_k}} \psi(t) dt. \]
Now we give the estimate for $I_4$. Similarly, we choose $1 < s < \infty$ such that $\frac{1}{s} = \frac{1}{2_1} + \frac{1}{2_2}$. Let
$$
\overline{b_{i,j}}(x) := \left| (b_i(x) - b_i,B) \left( b_j,B - b_j,\omega_j,\epsilon_j(t)B \right) \right|.
$$

Then Minkowski's inequality and Hölder's inequality show us that

\[
I_4 \leq \int_{[0,1]^n} \prod_{i,j=1,2} \left( \frac{1}{\omega(B)^{1+\lambda_p}} \int_B \left| \left( \prod_{k=1}^{2} \frac{1}{\omega_k(B)^{1+\lambda_k}} \int_{s_k(t)B} |f_k|(x) |^p \omega_k(x)dx \right) \right|^p \psi(t) \right)^{1/p} dt
\]

\[
\leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B \left| b_i(x) - b_i,B \right|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt
\]

\[
\leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt
\]

\[
\leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt
\]

From the estimates of $I_1$ and $I_3$, we deduce that

\[
I_4 \leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt
\]

It can be deduced from the estimates of $I_1, I_2, I_3, I_4, I_5$ that

\[
I_5 \leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt,
\]

\[
I_6 \leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt.
\]

Combining the estimates of $I_1, I_2, I_3, I_4, I_5$ and $I_6$ gives

\[
\left( \frac{1}{\omega(B)^{1+\lambda_p}} \int_B \left| T^{2,n,b} (f_1, f_2)(x) \right|^p \omega(x)dx \right)^{1/p} \leq \prod_{i,j=1,2} \left( \frac{1}{\omega_i(B)} \int_B |b_i(x) - b_i,B|^s \omega_i(x)dx \right)^{1/s} \left| b_j,B - b_j,\omega_j,\epsilon_j(t)B \right| \psi(t) dt.
\]

This proves (i).
Now we prove the necessity in (ii). Assume that \( U_{2, n, b}^{2, n, b} \) is bounded from \( \dot{B}^{p_1, \lambda_1}_W (\mathbb{R}^d) \times \dot{B}^{p_2, \lambda_2}_W (\mathbb{R}^d) \) to \( \dot{B}^{p, \lambda}_W (\mathbb{R}^d) \). First, we need the following lemmas.

**Lemma 4.4** ([7, Lemma 2.3]) If \( \omega \in W_\alpha \) and has doubling property, then \( \log |x| \in BMO(\omega) \).

**Lemma 4.5** Let \( \omega \in W_\alpha \), where \( \alpha > -d, 1 < p < \infty \) and \( -\frac{1}{p} < \lambda \). Then the function \( f_0(x) = |x|^{(d + \alpha)_q} \) belongs to \( \dot{B}^{p, \lambda}_W \) and

\[
\|f\|_{\dot{B}^{p, \lambda}_W} = \left( \omega(S_d) \right)^{-\lambda} \left( \frac{1}{(d + \alpha)(1 + \lambda p)} \right)^{1/p}.
\]

Since the proof of Lemma 4.5 is straightforward, we omit it. Now we set \( b_1(x) = b_2(x) = \log |x| \). Lemma 4.4 gives \( b_1, b_2 \) belong to \( BMO_\omega (\mathbb{R}^d) \), thus \( b_1, b_2 \in CMO_{W_\alpha} (\mathbb{R}^d) \). Define \( f_k(x) = |x|^{(d + \alpha_k)\lambda} \) if \( x \in \mathbb{R}^d \setminus \{0\} \) and \( f_k(0) = 0 \). From Lemma 4.5, we get that

\[
\|f_k\|_{\dot{B}^{p_k, \lambda_k}_W} = \left( \omega_k(S_d) \right)^{-\lambda_k} \left( \frac{1}{(d + \alpha_k)(1 + \lambda p)} \right)^{1/p_k}.
\]

Also we have

\[
U_{2, n, b}^{2, n, b} (f_1, f_2) (x) = \prod_{k=1}^{2} |x|^{(d + \alpha_k)\lambda_k} \int_{[0,1]^n} \prod_{k=1}^{2} \left( |s_k(t)|^{(d + \alpha_k)\lambda_k} \log \frac{1}{|s_k(t)|} \right) dt.
\]

Let \( B = B(0, R) \) be any ball of \( \mathbb{R}^d \), then

\[
\left( \frac{1}{\omega(B)^{1 + \lambda p}} \int_B |U_{2, n, b}^{2, n, b} (f_1, f_2) (x)|^p \omega(x) dx \right)^{1/p} \leq \left( \frac{1}{\omega(B)^{1 + \lambda p}} \int \left| x \right|^{(d + \alpha)\lambda p} \omega(x) dx \right)^{1/p} \left( \int_{[0,1]^n} \prod_{k=1}^{2} \left( |s_k(t)|^{(d + \alpha_k)\lambda_k} \log \frac{1}{|s_k(t)|} \right) \psi(t) dt \right)
\]

\[
= \left( \omega(S_d) \right)^{-\lambda} \left( \frac{1}{(d + \alpha)(1 + \lambda p)} \right)^{1/p} \left( \int_{[0,1]^n} \prod_{k=1}^{2} \left( |s_k(t)|^{(d + \alpha_k)\lambda_k} \log \frac{1}{|s_k(t)|} \right) \psi(t) dt \right)
\]

\[
= \prod_{k=1}^{2} \left( \int_{[0,1]^n} \prod_{k=1}^{2} \left( |s_k(t)|^{(d + \alpha_k)\lambda_k} \log \frac{1}{|s_k(t)|} \right) \psi(t) dt \right).
\]

Taking the supremum over \( R > 0 \), we get

\[
\int_{[0,1]^n} \prod_{k=1}^{2} \left( |s_k(t)|^{(d + \alpha_k)\lambda_k} \log \frac{1}{|s_k(t)|} \right) \psi(t) dt < \infty,
\]

which ends the proof of Theorem 4.2.

\[\Box\]

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