On the product of functions in \( BMO \) and \( H^1 \) over spaces of homogeneous type

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**ARTICLE INFO**

*Article history:*
Received 2 July 2014
Available online 29 December 2014
Submitted by P. Koskela

**Keywords:**
Musielak–Orlicz function
Hardy space
\( BMO \)
Space of homogeneous type
Admissible function
Atomic decomposition

**ABSTRACT**

Let \( X \) be an RD-space, which means that \( X \) is a space of homogeneous type in the sense of Coifman–Weiss with the additional property that a reverse doubling property holds in \( X \). The aim of the present paper is to study the product of functions in \( BMO \) and \( H^1 \) in this setting. Our results generalize some recent results in [4] and [11].

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1. Introduction and statement of main results

A famous result of C. Fefferman states that \( BMO(\mathbb{R}^n) \) is the dual space of \( H^1(\mathbb{R}^n) \). Although, for \( f \in BMO(\mathbb{R}^n) \) and \( g \in H^1(\mathbb{R}^n) \), the point-wise product \( fg \) may not be an integrable function, one (see [2]) can view the product of \( f \) and \( g \) as a distribution, denoted by \( f \times g \). Such a distribution can be written as the sum of an integrable function and a distribution in a new Hardy space, so-called Hardy space of Musielak–Orlicz type (see [1,9]). A complete study about the product of functions in \( BMO \) and \( H^1 \) has been firstly done by Bonami, Iwaniec, Jones and Zinsmeister [2]. Recently, Li and Peng [11] generalized this study to the setting of Hardy and \( BMO \) spaces associated with Schrödinger operators. In particular, Li and Peng showed that if \( L = -\Delta + V \) is a Schrödinger operator with the potential \( V \) belongs to the reverse Hölder class \( RH_q \) for some \( q \geq n/2 \), then one can view the product of \( b \in BMO_L(\mathbb{R}^n) \) and \( f \in H^1_L(\mathbb{R}^n) \) as a distribution \( b \times f \) which can be written as the sum of an integrable function and a distribution in \( H^1_L(\mathbb{R}^n, d\mu) \). Here \( H^1_L(\mathbb{R}^n, d\mu) \) is the weighted Hardy–Orlicz space associated with \( L \), related to the Orlicz

© This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.31.

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http://dx.doi.org/10.1016/j.jmaa.2014.12.057

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function \( \varphi(t) = t/\log(e + t) \) and the weight \( d\nu(x) = dx/\log(e + |x|) \). More precisely, they proved the following.

**Theorem A.** For each \( f \in H^1_L(\mathbb{R}^n) \), there exist two bounded linear operators \( \mathcal{L}_f : BMO_L(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) and \( \mathcal{H}_f : BMO_L(\mathbb{R}^n) \to H^p_L(\mathbb{R}^n, d\nu) \) such that for every \( b \in BMO_L(\mathbb{R}^n) \),

\[
 b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).
\]

Let \( (\mathcal{X}, d, \mu) \) be a space of homogeneous type in the sense of Coifman–Weiss. Following Han, Müller and Yang [8], we say that \( (\mathcal{X}, d, \mu) \) is an RD-space if \( \mu \) satisfies reverse doubling property, i.e., there exists a constant \( C > 1 \) such that for all \( x \in \mathcal{X} \) and \( r \in (0, \frac{\text{diam}(\mathcal{X})}{2}) \),

\[
 \mu(B(x, 2r)) \geq C \mu(B(x, r)),
\]

where \( \text{diam}(\mathcal{X}) := \sup_{x,y \in \mathcal{X}} d(x, y) \). A typical example for such RD-spaces is the Carnot–Carathéodory space with doubling measure. We refer to the seminal paper of Han, Müller and Yang [8] (see also [5,6,13, 14]) for a systematic study of the theory of function spaces in harmonic analysis on RD-spaces.

Let \( (\mathcal{X}, d, \mu) \) be an RD-space. Recently, in analogy with the classical result of Bonami–Iwaniec–Jones–Zinsmeister, Feuto proved in [4] that:

**Theorem B.** For each \( f \in H^1(\mathcal{X}) \), there exist two bounded linear operators \( \mathcal{L}_f : BMO(\mathcal{X}) \to L^1(\mathcal{X}) \) and \( \mathcal{H}_f : BMO(\mathcal{X}) \to H^p(\mathcal{X}, d\nu) \) such that for every \( b \in BMO(\mathcal{X}) \),

\[
 b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).
\]

Here the weight \( d\nu(x) = d\mu(x)/\log(e+d(x_0, x)) \) with \( x_0 \in \mathcal{X} \) and the Orlicz function \( \varphi \) is as in Theorem A. It should be pointed out that in [4], for \( f = \sum_{j=1}^{\infty} \lambda_j a_j \), the author defined the distribution \( b \times f \) as

\[
 b \times f := \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j + \sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j \quad (1.1)
\]

by proving that the second series is convergent in \( H^p(\mathcal{X}, d\nu) \). This is made possible by the fact that \( H^p(\mathcal{X}, d\nu) \) is complete and is continuously imbedded into the space of distributions \( (G_0(\beta, \gamma))' \) (see Section 2), which is not established in [4]. Moreover one has to prove that definition (1.1) does not depend on the atomic decomposition of \( f \). In this paper, we give a definition for the distribution \( b \times f \) (see Section 3) which is similar to that of Bonami–Iwaniec–Jones–Zinsmeister.

Our first main result can be read as follows.

**Theorem 1.1.** For each \( f \in H^1(\mathcal{X}) \), there exist two bounded linear operators \( \mathcal{L}_f : BMO(\mathcal{X}) \to L^1(\mathcal{X}) \) and \( \mathcal{H}_f : BMO(\mathcal{X}) \to H^{\log}(\mathcal{X}) \) such that for every \( b \in BMO(\mathcal{X}) \),

\[
 b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).
\]

Here \( H^{\log}(\mathcal{X}) \) is the Musielak–Orlicz Hardy space related to the Musielak–Orlicz function \( \varphi(x, t) = \log(e+d(x_0, x)) + \log(e+tj) \) (see Section 2). Theorem 1.1 is an improvement of Theorem B since \( H^{\log}(\mathcal{X}) \) is a proper subspace of \( H^p(\mathcal{X}, d\nu) \).

Let \( \rho \) be an admissible function (see Section 2). Recently, Yang and Zhou [13,14] introduced and studied Hardy spaces and Morrey–Campanato spaces related to the function \( \rho \). There, they established that
$BMO_\rho(X)$ is the dual space of $H^1_0(X)$. Similar to the classical case, we can define the product of functions $b \in BMO_\rho(X)$ and $f \in H^1_0(X)$ as distributions $b \times f \in (G^0_0(\beta, \gamma))'$.

Our next main result is as follows.

**Theorem 1.2.** For each $f \in H^1_0(X)$, there exist two bounded linear operators $\mathcal{L}_{p,f} : BMO_\rho(X) \to L^1(X)$ and $\mathcal{H}_{p,f} : BMO_\rho(X) \to H^{log}(X)$ such that for every $b \in BMO_\rho(X)$,

$$b \times f = \mathcal{L}_{p,f}(b) + \mathcal{H}_{p,f}(b).$$

When $X \equiv \mathbb{R}^n$, $n \geq 3$, and $\rho(x) \equiv \sup \{r > 0 : \frac{1}{r^2} \int_{B(x,r)} V(y)dy \leq 1\}$, where $L = -\Delta + V$ is as in Theorem A, one has $BMO_\rho(X) \equiv BMO_\rho(\mathbb{R}^n)$ and $H^1_0(X) \equiv H^1_0(\mathbb{R}^n)$. So, Theorem 1.2 is an improvement of Theorem A since $H^{log}(\mathbb{R}^n)$ is a proper subspace of $H^1_0(\mathbb{R}^n, d\nu)$ (see [10]).

The following conjecture is suggested by A. Bonami and F. Bernicot.

**Conjecture.** There exist two bounded bilinear operators $\mathcal{L} : BMO(X) \times H^1(X) \to L^1(X)$ and $\mathcal{H} : BMO(X) \times H^1(X) \to H^{log}(X)$ such that

$$b \times f = \mathcal{L}(b, f) + \mathcal{H}(b, f).$$

It should be pointed out that when $X = \mathbb{R}^n$ and $H^{log}(X)$ is replaced by $H^0(\mathbb{R}^n, d\nu)$, the above conjecture is just Conjecture 1.7 of [2], which has been answered recently by Bonami, Grellier and Ky [1] (see also [10]).

Throughout the whole paper, $C$ denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. We write $f \sim g$ if there exists a constant $C > 1$ such that $C^{-1} f \leq g \leq Cf$.

The paper is organized as follows. In Section 2, we present some notations and preliminaries about $BMO$ type spaces and Hardy type spaces on RD-spaces. Section 3 is devoted to prove Theorem 1.1. Finally, we give the proof for Theorem 1.2 in Section 4.

2. Preliminaries

Let $d$ be a quasi-metric on a set $X$, that is, $d$ is a nonnegative function on $X \times X$ satisfying

(a) $d(x,y) = d(y,x)$,
(b) $d(x,y) > 0$ if and only if $x \neq y$,
(c) there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in X$, $d(x,z) \leq \kappa(d(x,y) + d(y,z)).$

A triple $(X, d, \mu)$ is called a space of homogeneous type in the sense of Coifman–Weiss [3] if $\mu$ is a regular Borel measure satisfying doubling property, i.e. there exists a constant $C > 1$ such that for all $x \in X$ and $r > 0$, $\mu(B(x, 2r)) \leq C \mu(B(x, r))$.

Following Han, Müller and Yang [8], $(X, d, \mu)$ is called an RD-space if $(X, d, \mu)$ is a space of homogeneous type and $\mu$ also satisfies reverse doubling property, i.e. there exists a constant $C > 1$ such that $\mu(B(x, 2r)) \geq C \mu(B(x, r))$ for all $x \in X$ and $r \in (0, \frac{\text{diam}(X)}{2})$, where $\text{diam}(X) = \sup_{x,y \in X} d(x,y)$.
It should be pointed out that \((\mathcal{X}, d, \mu)\) is an RD-space if and only if there exist constants \(0 < \delta \leq n\) and \(C > 1\) such that for all \(x \in \mathcal{X}, 0 < r < \text{diam}(\mathcal{X})/2\), and \(1 \leq \lambda < \text{diam}(\mathcal{X})/(2r),\)

\[
C^{-1}\lambda^q \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^q \mu(B(x, r)). \tag{2.2}
\]

In what follows, for \(x, y \in \mathcal{X}\) and \(r > 0\), we denote \(V_r(x) := \mu(B(x, r))\) and \(V(x, y) := \mu(B(x, d(x, y)))\).

Next we wish to recall the space of test functions introduced originally by Han, Müller and Yang [7, Definition 2.2] (see also [8, Definition 2.8]).

**Definition 2.1.** Let \(x_0 \in \mathcal{X}, r > 0, 0 < \beta \leq 1\) and \(\gamma > 0\). A function \(f\) is said to belong to the space of test functions, \(\mathcal{G}(x_0, r, \beta, \gamma)\), if there exists a positive constant \(C_f\) such that

(i) \(|f(x)| \leq C_f \frac{1}{V_{r+d(x_0, x)}(x)} \frac{r}{r+d(x_0, x)} \gamma \) for all \(x \in \mathcal{X}\);

(ii) \(|f(x) - f(y)| \leq C_f \frac{1}{V_{r+d(x_0, x)}(x)} \frac{r}{r+d(x_0, x)} \gamma \) for all \(x, y \in \mathcal{X}\) satisfying that \(d(x, y) \leq \frac{r}{2}\).

For any \(f \in \mathcal{G}(x_0, r, \beta, \gamma)\), we define

\[
\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} := \inf\{C_f : \text{(i) and (ii) hold}\}.
\]

Let \(\rho\) be a positive function on \(\mathcal{X}\). Following Yang and Zhou [14], the function \(\rho\) is said to be admissible if there exist positive constants \(C_0\) and \(k_0\) such that for all \(x, y \in \mathcal{X}\),

\[
\rho(y) \leq C_0 \left[ \rho(x) \right]^{1/(1+k_0)} \left[ \rho(x) + d(x, y) \right]^{k_0/(1+k_0)}.
\]

*Throughout the whole paper*, we always assume that \(\mathcal{X}\) is an RD-space with \(\mu(\mathcal{X}) = \infty\), and \(\rho\) is an admissible function on \(\mathcal{X}\). Also we fix \(x_0 \in \mathcal{X}\).

In **Definition 2.1**, it is easy to see that \(\mathcal{G}(x_0, 1, \beta, \gamma)\) is a Banach space. For simplicity, we write \(\mathcal{G}(\beta, \gamma)\) instead of \(\mathcal{G}(x_0, 1, \beta, \gamma)\). Let \(\epsilon \in (0, 1]\) and \(\beta, \gamma \in (0, \epsilon]\), we define the space \(\mathcal{G}_0(\beta, \gamma)\) to be the completion of \(\mathcal{G}(\epsilon, \epsilon)\) in \(\mathcal{G}(\beta, \gamma)\), and denote \((\mathcal{G}_0(\beta, \gamma))'\) the space of all continuous linear functionals on \(\mathcal{G}_0(\beta, \gamma)\). We say that \(f\) is a distribution if \(f\) belongs to \((\mathcal{G}_0(\beta, \gamma))'\).

Remark that, for any \(x \in \mathcal{X}\) and \(r > 0\), one has \(\mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)\) with equivalent norms, but of course the constants are depending on \(x\) and \(r\).

Let \(f\) be a distribution in \((\mathcal{G}_0(\beta, \gamma))'\). We define the grand maximal functions \(\mathcal{M}(f)\) and \(\mathcal{M}_\rho(f)\) as following

\[
\mathcal{M}(f)(x) := \sup \left\{ \|f, \varphi\| : \varphi \in \mathcal{G}_0(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0 \right\},
\]

\[
\mathcal{M}_\rho(f)(x) := \sup \left\{ \|f, \varphi\| : \varphi \in \mathcal{G}_0(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r \in (0, \rho(x)) \right\}.
\]

Let \(L^{\log}(\mathcal{X})\) (see [1,9] for details) be the Musielak–Orlicz type space of \(\mu\)-measurable functions \(f\) such that

\[
\int_{\mathcal{X}} \frac{|f(x)|}{\log(e + |f(x)|) + \log(e + d(x, 0, x))} \mu(x) < \infty.
\]

For \(f \in L^{\log}(\mathcal{X})\), we define the “norm” of \(f\) as

\[
\|f\|_{L^{\log}} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{|f(x)|^\lambda}{\log(e + |f(x)|) + \log(e + d(x, 0, x))} \mu(x) \leq 1 \right\}.
\]
**Definition 2.2.** Let $\epsilon \in (0, 1)$ and $\beta, \gamma \in (0, \epsilon)$.

(i) The Hardy space $H^1(\mathcal{X})$ is defined by

$$H^1(\mathcal{X}) = \{ f \in (G'_0(\beta, \gamma))' : \|f\|_{H^1} := \|\mathcal{M}(f)\|_{L^1} < \infty \}.$$  

(ii) The Hardy space $H^1_{\rho}(\mathcal{X})$ is defined by

$$H^1_{\rho}(\mathcal{X}) = \{ f \in (G'_0(\beta, \gamma))' : \|f\|_{H^1_{\rho}} := \|\mathcal{M}_{\rho}(f)\|_{L^1} < \infty \}.$$  

(iii) The Hardy space $H^{\log}(\mathcal{X})$ is defined by

$$H^{\log}(\mathcal{X}) = \{ f \in (G'_0(\beta, \gamma))' : \|f\|_{H^{\log}} := \|\mathcal{M}(f)\|_{L^{\log}} < \infty \}.$$  

It is clear that $H^1(\mathcal{X}) \subset H^1_{\rho}(\mathcal{X})$ and $H^1(\mathcal{X}) \subset H^{\log}(\mathcal{X})$ with the inclusions are continuous. It should be pointed out that the Musielak–Orlicz Hardy space $H^{\log}(\mathcal{X})$ is a proper subspace of the weighted Hardy–Orlicz space $H^{\nu}(\mathcal{X}, \nu)$ studied in [4]. We refer to [9] for an introduction to Musielak–Orlicz Hardy spaces on the Euclidean space $\mathbb{R}^n$.

**Definition 2.3.** Let $q \in (1, \infty]$.

(i) A measurable function $a$ is called an $(H^1, q)$-atom related to the ball $B(x, r)$ if

- (a) $\text{supp } a \subset B(x, r)$,
- (b) $\|a\|_{L^q} \leq (V_r(x))^{1/q-1}$,
- (c) $\int_B a(y) d\mu(y) = 0$.

(ii) A measurable function $a$ is called an $(H^1_{\rho}, q)$-atom related to the ball $B(x, r)$ if $r \leq 2\rho(x)$ and $a$ satisfies

- (a) and (b), and when $r < \rho(x)$, $a$ also satisfies (c).

The following results were established in [5,14].

**Theorem 2.1.** Let $q \in (1, \infty]$. Then, we have:

(i) The space $H^1(\mathcal{X})$ coincides with the Hardy space $H^{1,q}_{\text{st}}(\mathcal{X})$ of Coifman–Weiss. More precisely, $f \in H^1(\mathcal{X})$ if and only if $f$ can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the $a_j$’s are $(H^1, q)$-atoms and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$. Moreover,

$$\|f\|_{H^1} \sim \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$  

(ii) $f \in H^1_{\rho}(\mathcal{X})$ if and only if $f$ can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the $a_j$’s are $(H^1_{\rho}, q)$-atoms and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$. Moreover,

$$\|f\|_{H^1_{\rho}} \sim \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$  

In what follows, for any ball $B \subset \mathcal{X}$ and $g \in L^1_{\text{loc}}(\mathcal{X})$, we denote by $g_B$ the average value of $g$ over the ball $B$ and denote
MO(g, B) := \frac{1}{\mu(B)} \int_B |g(x) - g_B| d\mu(x).

Recall (see [3]) that a function \( f \in L^1_{\text{loc}}(\mathcal{X}) \) is said to be in \( BMO(\mathcal{X}) \) if

\[ \| f \|_{BMO} = \sup_B MO(f, B) < \infty, \]

where the supremum is taken all over balls \( B \subset \mathcal{X} \).

**Definition 2.4.** Let \( \rho \) be an admissible function and \( D := \{ B(x, r) \subset \mathcal{X} : r \geq \rho(x) \} \). A function \( f \in L^1_{\text{loc}}(\mathcal{X}) \) is said to be in \( BMO_{\rho}(\mathcal{X}) \) if

\[ \| f \|_{BMO_{\rho}} = \| f \|_{BMO} + \sup_{B \in D} \frac{1}{\mu(B)} \int_B |f(x)| d\mu(x) < \infty. \]

The following results are well-known; see [3,5,13].

**Theorem 2.2.**

(i) The space \( BMO(\mathcal{X}) \) is the dual space of \( H^1(\mathcal{X}) \).

(ii) The space \( BMO_{\rho}(\mathcal{X}) \) is the dual space of \( H^1_{\rho}(\mathcal{X}) \).

3. The product of functions in \( BMO(\mathcal{X}) \) and \( H^1(\mathcal{X}) \)

Remark that if \( g \in \mathcal{G}(\beta, \gamma) \), then

\[ \| g \|_{L^\infty} \leq C \frac{1}{V_1(x_0)} \| g \|_{\mathcal{G}(\beta, \gamma)} \]  \hspace{1cm} (3.1)

and

\[ \| g \|_{L^1} \leq \left( C + \sum_{j=0}^{\infty} 2^{-j\gamma} \right) \| g \|_{\mathcal{G}(\beta, \gamma)} \leq C \| g \|_{\mathcal{G}(\beta, \gamma)}. \]  \hspace{1cm} (3.2)

**Proposition 3.1.** Let \( \beta \in (0, 1] \) and \( \gamma \in (0, \infty) \). Then \( g \) is a pointwise multiplier of \( BMO(\mathcal{X}) \) for all \( g \in \mathcal{G}(\beta, \gamma) \). More precisely,

\[ \| gf \|_{BMO} \leq C \frac{1}{V_1(x_0)} \| g \|_{\mathcal{G}(\beta, \gamma)} \| f \|_{BMO^+} \]

for all \( f \in BMO(\mathcal{X}) \). Here and what in follows,

\[ \| f \|_{BMO^+} := \| f \|_{BMO} + \frac{1}{V_1(x_0)} \int_{B(x_0, 1)} |f(x)| d\mu(x). \]

Using Proposition 3.1, for \( b \in BMO(\mathcal{X}) \) and \( f \in H^1(\mathcal{X}) \), one can define the distribution \( b \times f \in (\mathcal{G}_0^*(\beta, \gamma))^\prime \) by the rule

\[ \langle b \times f, \phi \rangle := \langle \phi b, f \rangle \]  \hspace{1cm} (3.3)
for all \( \phi \in \mathcal{G}_0(\beta, \gamma) \), where the second bracket stands for the duality bracket between \( H^1(\mathcal{X}) \) and its dual \( BMO(\mathcal{X}) \).

**Proof of Proposition 3.1.** By (3.1) and the pointwise multipliers characterization of \( BMO(\mathcal{X}) \) (see [12, Theorem 1.1]), it is sufficient to show that

\[
\log(e + 1/r)MO(g, B(a, r)) \leq C \frac{1}{V_1(x_0)}\|g\|_{\mathcal{G}(\beta, \gamma)} \tag{3.4}
\]

and

\[
\log(e + d(x_0, a) + r)MO(g, B(a, r)) \leq C \frac{1}{V_1(x_0)}\|g\|_{\mathcal{G}(\beta, \gamma)} \tag{3.5}
\]

hold for all balls \( B(a, r) \subset \mathcal{X} \). It is easy to see that (3.4) follows from (3.1) and the Lipschitz property of \( g \) (see (ii) of Definition 2.1). Let us now establish (3.5). If \( r < 1 \), then (3.5) follows from the Lipschitz property of \( g \) and the fact that \( \lim_{\lambda \to \infty} \frac{\log(\lambda)}{\lambda} = 0 \). Otherwise, we consider the following two cases:

(a) The case: \( 1 \leq r \leq \frac{1}{\kappa(4\kappa^2 + 1)}d(x_0, a) \). Then, for every \( x, y \in B(a, r) \), one has \( d(x_0, a) \leq \frac{4\kappa^2 + 1}{4\kappa}d(x, a) \) and \( d(x, y) < \frac{d(x_0, x)}{2\kappa} \). Hence, the Lipschitz property of \( g \) yields

\[
|g(x) - g(y)| \leq C\|g\|_{\mathcal{G}(\beta, \gamma)} \frac{1}{V_1(x_0)}\left(\frac{1}{d(x_0, a)}\right)^\gamma.
\]

This implies that (3.5) holds since \( \lim_{\lambda \to \infty} \frac{\log(\lambda)}{\lambda^\gamma} = 0 \).

(b) The case: \( r > \frac{1}{\kappa(4\kappa^2 + 1)}d(x_0, a) \) and \( r \geq 1 \). Then, one has \( B(x_0, r) \subset B(a, \kappa(4\kappa^3 + \kappa + 1)r) \). Hence, by (2.2), we obtain

\[
\log(e + d(x_0, a) + r)MO(g, B(a, r)) \leq C \frac{\log(2r)}{V_r(x_0)} \|g\|_{L^1} \leq C \frac{\log(2r)}{r^\kappa} \frac{1}{V_1(x_0)}\|g\|_{\mathcal{G}(\beta, \gamma)}
\]

\[
\leq C \frac{1}{V_1(x_0)}\|g\|_{\mathcal{G}(\beta, \gamma)}.
\]

This proves (3.5) and thus the proof of Proposition 3.1 is finished. \( \square \)

Next we define \( L^\Xi(\mathcal{X}) \) as the space of \( \mu \)-measurable functions \( f \) such that

\[
\int_{\mathcal{X}} \frac{e^{|f(x)|} - 1}{(1 + d(x_0, x))^{2(n+1)}}d\mu(x) < \infty.
\]

Then, the norm on the space \( L^\Xi(\mathcal{X}) \) is defined by

\[
\|f\|_{L^\Xi} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{e^{|f(x)|/\lambda} - 1}{(1 + d(x_0, x))^{2(n+1)}}d\mu(x) \leq 1 \right\}.
\]

Recall the following two lemmas due to Feuto [4].

**Lemma 3.1.** For every \( f \in BMO(\mathcal{X}) \),

\[
\|f - f_{B(x_0, 1)}\|_{L^\Xi} \leq C\|f\|_{BMO}.
\]
Lemma 3.2. Let $q \in (1, \infty]$. Then,
\[
\| (b - b_B)\mathcal{M}(a) \|_{L^1} \leq C \|b\|_{BMO}
\]
for all $b \in BMO(X)$ and for all $(H^1, q)$-atom $a$ related to the ball $B$.

The main point in the proof of Theorem 1.1 is the following.

Proposition 3.2.

(i) For any $f \in L^1(X)$ and $g \in L^{\infty}(X)$, we have
\[
\|fg\|_{L^{\log}} \leq 64(n + 1)^2 \|f\|_{L^1} \|g\|_{L^{\infty}}.
\]

(ii) For any $f \in L^1(X)$ and $g \in BMO(X)$, we have
\[
\|fg\|_{L^{\log}} \leq C \|f\|_{L^1} \|g\|_{BMO^+}.
\]

Proof. (i) If $\|f\|_{L^1} = 0$ or $\|g\|_{L^{\infty}} = 0$, then there is nothing to prove. Otherwise, we may assume that $\|f\|_{L^1} = \|g\|_{L^{\infty}} = \frac{1}{8(n+1)}$ since homogeneity of the norms. Then, we need to prove that
\[
\int_X \frac{|f(x)g(x)|}{\log(e + |f(x)g(x)|) + \log(e + d(x,0))} \, d\mu(x) \leq 1.
\]

Indeed, by using the following two inequalities
\[
\log(e + ab) \leq 2(\log(e + a) + \log(e + b)), \quad a, b \geq 0,
\]
and
\[
\frac{ab}{\log(e + ab)} \leq a + (e^b - 1), \quad a, b \geq 0,
\]
we obtain that, for every $x \in X$,
\[
\frac{(1 + d(x_0, x))^{2(n+1)} |f(x)g(x)|}{4(n + 1)(\log(e + |f(x)g(x)|) + \log(e + d(x,0)))} \leq \frac{(1 + d(x_0, x))^{2(n+1)} |f(x)g(x)|}{2(\log(e + |f(x)g(x)|) + \log(e + (1 + d(x,0))^{2(n+1)}))} \leq \frac{(1 + d(x_0, x))^{2(n+1)} |f(x)||g(x)|}{\log(e + (1 + d(x,0))^{2(n+1)}|f(x)||g(x)|) \leq (1 + d(x_0, x))^{2(n+1)} |f(x)| + (e^{|g(x)|} - 1).
\]

This together with the fact $8(n + 1)(e^{|g(x)|} - 1) \leq e^{8(n+1)|g(x)|} - 1$ give
\[
\int_X \frac{|f(x)g(x)|}{\log(e + |f(x)g(x)|) + \log(e + d(x,0))} \, d\mu(x)
\]
\[
\leq 4(n + 1)\|f\|_{L^1} + \frac{1}{2} \int_X \frac{e^{8(n+1)|g(x)|} - 1}{(1 + d(x,0))^{2(n+1)}} \, d\mu(x) \leq \frac{1}{2} + \frac{1}{2} = 1,
\]
which completes the proof of (i).

(ii) It follows directly from (i) and Lemma 3.1. □
Now we are ready to give the proof for Theorem 1.1.

**Proof of Theorem 1.1.** By (i) of Theorem 2.1, \( f \) can be written as

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j
\]

where the \( a_j \)'s are \((H^1, \infty)\)-atoms related to the balls \( B_j \)'s and \( \sum_{j=1}^{\infty} |\lambda_j| \leq C \|f\|_{H^1} \). Therefore, for all \( b \in BMO(\mathcal{X}) \), we have

\[
\left\| \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j \right\|_{L^1} \leq \sum_{j=1}^{\infty} |\lambda_j| \left\| (b - b_{B_j}) a_j \right\|_{L^1} \leq C \|b\|_{BMO} \|f\|_{H^1}.
\]

(3.6)

By this and (3.3), we see that the series \( \sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j \) converges to \( b \times f - \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j \) in \((G_0(\beta, \gamma))'\). Consequently, if we define the decomposition operators as

\[
\mathcal{L}_f(b) = \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j
\]

and

\[
\mathcal{H}_f(b) = \sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j,
\]

where the summations are in \((G_0(\beta, \gamma))'\), then it is clear that \( \mathcal{L}_f : BMO(\mathcal{X}) \to L^1(\mathcal{X}) \) is a bounded linear operator, since (3.6), and for every \( b \in BMO(\mathcal{X}) \),

\[
b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).
\]

Now we only need to prove that the distribution \( \mathcal{H}_f(b) \) is in \( H^{log}(\mathcal{X}) \). Indeed, by Lemma 3.2 and (ii) of Proposition 3.2, we get

\[
\left\| M(\mathcal{H}_f(b)) \right\|_{L^{log}} \leq \left\| \sum_{j=1}^{\infty} |\lambda_j| b_{B_j} M(a_j) \right\|_{L^{log}} \leq \left\| \sum_{j=1}^{\infty} |\lambda_j| (b - b_{B_j}) a_j \right\|_{L^1} + \left\| \sum_{j=1}^{\infty} |\lambda_j| a_j \right\|_{L^{log}} \leq C \|f\|_{H^1} \|b\|_{BMO^+}.
\]

This proves that \( \mathcal{H}_f \) is bounded from \( BMO(\mathcal{X}) \) into \( H^{log}(\mathcal{X}) \), and thus ends the proof of Theorem 1.1. \( \square \)

4. **The product of functions in \( BMO_\rho(\mathcal{X}) \) and \( H^1_\rho(\mathcal{X}) \)**

For \( f \in BMO_\rho(\mathcal{X}) \), a standard argument gives

\[
\|f\|_{BMO^+} \leq C \log(\rho(x_0) + 1/\rho(x_0)) \|f\|_{BMO_\rho}, \tag{4.1}
\]
Proposition 4.1. Let $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. Then, $g$ is a pointwise multiplier of $BMO_\rho(\mathcal{X})$ for all $g \in \mathcal{G}(\beta, \gamma)$. More precisely, for every $f \in BMO_\rho(\mathcal{X})$,

$$
\|gf\|_{BMO_\rho} \leq C \frac{\log(\rho(x_0) + 1/\rho(x_0))}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \|f\|_{BMO_\rho}.
$$

Proof. By Proposition 3.1, (4.1) and (3.1), we get

$$
\|gf\|_{BMO_\rho} \leq \|gf\|_{BMO} + \|g\|_{L^\infty} \sup_{B \in D} \frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x)
\leq C \frac{\log(\rho(x_0) + 1/\rho(x_0))}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \|f\|_{BMO_\rho}. \quad \square
$$

Using Proposition 4.1, for $b \in BMO_\rho(\mathcal{X})$ and $f \in H^1_\rho(\mathcal{X})$, one can define the distribution $b \times f \in (\mathcal{G}_0(\beta, \gamma))'$ by the rule

$$
(b \times f, \phi) := \langle \phi b, f \rangle \quad (4.2)
$$

for all $\phi \in \mathcal{G}_0(\beta, \gamma)$, where the second bracket stands for the duality bracket between $H^1_\rho(\mathcal{X})$ and its dual $BMO_\rho(\mathcal{X})$.

Proof of Theorem 1.2. By (ii) of Theorem 2.1, there exist a sequence of $(H^1_\rho, \infty)$-atoms $\{a_j\}^\infty_{j=1}$ related to the sequence of balls $\{B(x_j, r_j)\}^\infty_{j=1}$ and $\sum^\infty_{j=1} |\lambda_j| \leq C \|f\|_{H^1_\rho}$ such that

$$
f = \sum^\infty_{j=1} \lambda_j a_j = f_1 + f_2,
$$

where $f_1 = \sum_{r_j < \rho(x_j)} \lambda_j a_j \in H^1(\mathcal{X})$ and $f_2 = \sum_{r_j \geq \rho(x_j)} \lambda_j a_j$.

We define the decomposition operators as following

$$
\mathcal{L}_{p,f}(b) = \mathcal{L}_{f_1}(b) + bf_2
$$

and

$$
\mathcal{H}_{p,f}(b) = \mathcal{H}_{f_1}(b),
$$

where the operators $\mathcal{L}_{f_1}$ and $\mathcal{H}_{f_1}$ are as in Theorem 1.1. Then, Theorem 1.1 together with its proof and (4.1) give

$$
\|\mathcal{L}_{p,f}(b)\|_{L^1} \leq \|\mathcal{L}_{f_1}(b)\|_{L^1} + \sum_{r_j \geq \rho(x_j)} |\lambda_j| \|ba_j\|_{L^1}
\leq C \|f_1\|_{H^1} \|b\|_{BMO} + C \|b\|_{BMO_\rho} \sum_{r_j \geq \rho(x_j)} |\lambda_j|
\leq C \|f\|_{H^1_\rho} \|b\|_{BMO_\rho}
$$

and

$$
\|\mathcal{H}_{p,f}(b)\|_{H^{log}} \leq C \|f_1\|_{H^1} \|b\|_{BMO^+} \leq C \|f\|_{H^1_\rho} \|b\|_{BMO_\rho}.
$$
This proves that the linear operator $\mathcal{L}_{p,f} : BMO_p(\mathcal{X}) \to L^1(\mathcal{X})$ is bounded and the linear operator $\mathcal{H}_{p,f} : BMO_p(\mathcal{X}) \to H^\log(\mathcal{X})$ is bounded. Moreover,

$$b \times f = b \times f_1 + b \times f_2 = (\mathcal{L}_{f_1}(b) + \mathcal{H}_{f_1}(b)) + bf_2 = \mathcal{L}_{p,f}(b) + \mathcal{H}_{p,f}(b),$$

which ends the proof of Theorem 1.2. □

Acknowledgments

The author would like to thank Aline Bonami, Sandrine Grellier, Dachun Yang and Frédéric Bernicot for very useful suggestions.

References


